A technique for obtaining approximate solutions for a class of integral equations arising in radiative heat transfer

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Approximate but accurate solutions for a class of integral equations, such as those that arise in radiative transfer, are obtained with two-step method proposed. The first term of a Taylor series expansion is used to obtain a first-order approximation, followed by collocation with the first approximation as a trial function. This technique appears to be particularly attractive for non-uniform temperature distributions, for which direct numerical integration may be subject to instability, or require an excessive amount of computation, and for which other approximate methods may require numerous terms or an excessive amount of computation. Illustrative calculations for representative thermal problems indicate that the accuracy of the method depends critically on the location and number of collocation points and suggestions are provided for these choices

Keywords: *radiative heat transfer, approximate methods, collocation, configuration factors*

Radiative transfer between non-black surfaces is generally represented by integral equations as contrasted with integrals for blackbody radiation and differential equations for conduction and convection. The applicable techniques of solution are of course quite different.

Methods of solving integral equations in general have been reviewed by Squire[1] and others, and those which arise in radiative transfer between grey surfaces by Sparrow and Cess[2], Siegel and Howell[3] and others. Exact solutions in closed form are possible only for a few simple problems. Numerical solutions can usually be developed by successive iteration or by approximating the integral by a quadrature and hence reducing the problem to a set of algebraic equations. In principle, such numerical solutions can be carried out to any desired degree of accuracy. In practice, however, instability, slow convergence and excessive computation may limit either method. Hence, despite the continued advance in computers and numerical methods, approximate methods remain of interest and utility in this field.

A variety of approximate methods have been proposed. Approximation of the kernel of the integral by a finite series of exponentials, as first employed by Buckley[4], is helpful in many cases. The integral can then be eliminated by differentiating the integral equation twice for every exponential term. However, this transforms the integral equation to a differential equation and requires the manufacture of boundary conditions equal in number to the differentiations. Expansion of the integrand in Taylor's series also offers the possibility of reducing an integral equation to a differential equation by differentiation. Chu *et al*[5] showed that, if the Fourier transform of the kernel of the integral can be expressed or approximated by a ratio of power series of the variable of transformation, the integral equation can be reduced to a diffusion-type differential equation. They further derived an exact, if awkward, expression for the added boundary conditions.

Another class of approximations involves the representation of the integrand by a finite series, with the constants in the series evaluated by variational methods or by the method of weighted residuals. Sparrow and Haji-Sheikh[6] have illustrated the use of the latter method with orthogonal functions and least squares.

The accuracy of all of these numerical and approximate methods is limited by the well-known insensitivity of integrals to their kernel. Hence a simple method of obtaining an approximate and unique solution remains of interest; this paper describes such a method, giving examples chosen to have exact solutions available for comparison.

Radiant flux density between parallel strips

Uniform equal temperatures

Radiative transfer between two parallel strips at the same uniform temperature was chosen as a first example because this problem has been used as an illustration for most of the approximate methods which have previously been proposed, and because an exact solution is available for comparative purposes. The strips have a width w and are separated by the distance h (Fig 1). The temperature of the upper strip is T_1 and of the lower strip is T_2 $(=T_1)$.

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Received 11 April 1984 and accepted for publication in revised form on 6 September 1984

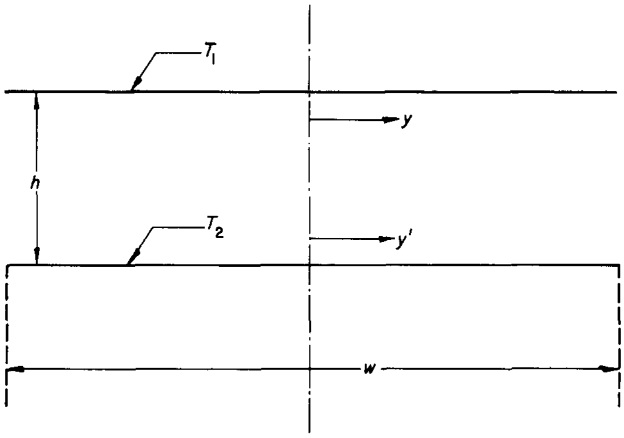

Fig 1 Model for radiative transfer between grey parallel strips

Distance to the right of the centre of the upper strip is designated by y and of the lower strip by y'. The exact formulation for the dimensionless radiosity in the form of an integral equation, as given by Sparrow[7], is:

$$\beta\{Y\}=\varepsilon+\frac{(1-\varepsilon)\gamma^2}{2}\int_{-1/2}^{1/2}\beta(\xi)G\{Y,\xi\}\mathrm{d}\xi \quad (1)$$

where:

$$G\{Y,Y'\}=1/[(Y-Y')^2+\gamma^2]^{3/2} \quad (2)$$

Eq (1) is a Fredholm integral equation of the second kind. The dimensionless radiosity defined by Eq (1) is the ratio of the radiant flux to the emission from a black body, whereas that used by Sparrow is the ratio of the radiant flux to the emission from a grey surface.

First-order approximation

Taking the first term of the Taylor series expansion of $\beta\{\xi\}$, namely $\beta\{Y\}$, permits taking that function outside the integral. Then, on rearrangement, Eq (1) becomes:

$$\beta^*\{Y\}\simeq\frac{\varepsilon}{1-\frac{(1-\varepsilon)\gamma^2}{2}\int_{-1/2}^{1/2}G\{Y,\xi\}\mathrm{d}\xi} \quad (3)$$

where here the asterisk indicates the first-order approximation. This approximation, which reduces solution of an integral equation to evaluation of an integral, was apparently first suggested by Wagner[8]. The remaining integral in Eq (3) can be evaluated analytically giving:

$$\beta^*\{Y\}=\frac{\varepsilon}{1-\frac{1-\varepsilon}{2}\left(\frac{\frac{1}{2}+Y}{((\frac{1}{2}+Y)^2+\gamma^2)^{1/2}}+\frac{\frac{1}{2}-Y}{((\frac{1}{2}-Y)^2+\gamma^2)^{1/2}}\right)} \quad (4)$$

Eq (4), labelled the one-term solution, is compared in Figs 2–4 with the exact numerical solution of Sparrow[7]

Notation

a_i	Arbitrary constants in trial functions
A	Arbitrary constant to be determined by collocation
b_i	Arbitrary constants in trial functions
B	Arbitrary constant to be determined by collocation
$B\{X\}$	Radiosity, total radiant flux density from cylindrical surface at X
$B\{Y\}$	Radiosity, total radiant flux density from strip at Y
c_i	Arbitrary constants in trial functions
C	Arbitrary constant to be determined by collocation
d	Tube diameter
$F\{X\}$	View factor from cylindrical ring at X to end of cylinder at $X=0$
$G\{Y,Y'\}$	View factor for radiant interchange between locations Y and Y'
h	Distance between parallel strips
$I\{L,X\}$	Integral in Eq (22)
$K\{X,X'\}$	View factor for radiant interchange between differential rings at X and X'
l	Length of cylinder
L	Dimensionless length of cylinder, l/d
$T\{X\}$	Absolute temperature along cylindrical wall
T_1	Absolute temperature on upper parallel plate and on disc at $X=0$
T_2	Absolute temperature on lower parallel plate and on disc at $X=L$
w	Width of strip
x	Distance along cylinder from disc at T_1
X	x/d
y	Distance from centre line of upper strip
y'	Distance from centre line of lower strip
Y	y/w
Y'	y'/w
$\beta\{X\}$	Dimensionless radiosity on surface of cylinder, $B\{X\}/\sigma T_1^4$
$\beta\{Y\}$	Dimensionless radiosity on strip, $B\{Y\}/\sigma T^4$
$\beta'\{Y\}$	Dimensionless radiosity, Eq (14)
$\beta''\{Y\}$	Dimensionless radiosity, Eq (16)
β^*	First-order approximation for dimensionless radiosity, as obtained from first term of Taylor expansion
β^{**}	Second-order approximation for dimensionless radiosity, as obtained by collocation
γ	h/w
ε	Emissivity
$\theta\{X\}$	Specified dimensionless temperature distribution, $T\{X\}/T_1$
θ_2	Dimensionless temperature of end of cylinder at $x=l$, T_2/T_1
ξ	Dummy variable
σ	Stefan-Boltzmann constant

Subscripts

1	Upper strip or disc at $X=0$
2	Lower strip or disc at $X=L$

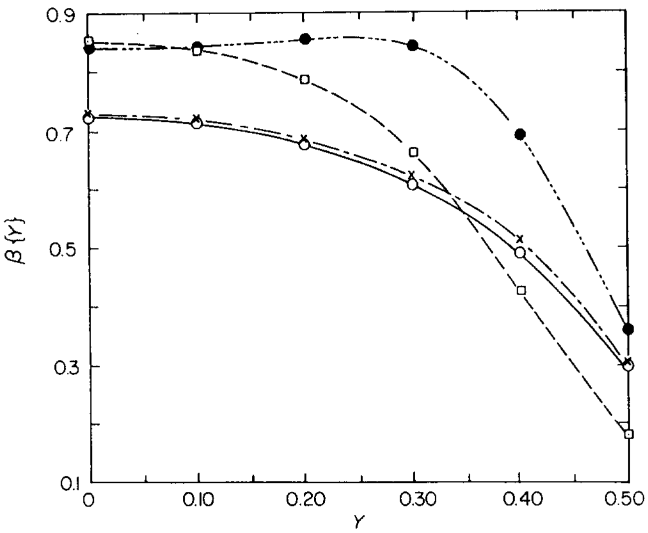

Fig 2 Comparison of approximations and exact solution for dimensionless radiosity on grey parallel strips at equal uniform temperature with $\varepsilon=0.1$ and $\gamma=0.1$. (○ Exact; □ One-term expansion, Eq (4); ● Two-term expansion, Eq (9) with Y=0, 0.5; × Three-term expansion, Eq (9) Y=0, 0.25, 0.5)

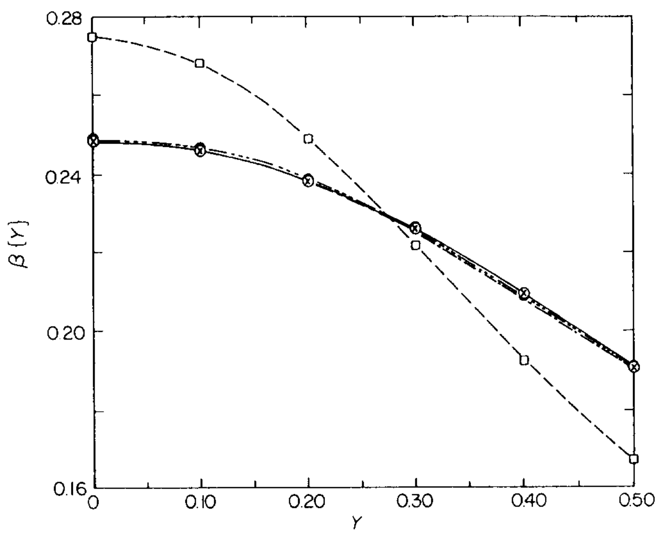

Fig 3 Comparison of approximations and exact solution for dimensionless radiosity on grey parallel strips at equal uniform temperature with $\varepsilon=0.1$ and $\gamma=0.5$. (Symbols as in Fig 2)

for $\varepsilon=0.1$ and $\gamma=0.1$, 0.5 and 1, respectively. This approximation is seen to have only qualitative value.

Second-order approximation

A greatly improved approximation can be obtained by collocation, using the first-order approximation as the trial function, ie by letting:

$$\beta^{**}\{Y\}=\phi_1\{Y\}+a_2\phi_2\{Y\}+a_3\phi_3\{Y\}+\dots \tag{5}$$

where:

$$\phi_1\{Y\}=\beta^*\{Y\} \tag{6}$$

$$\phi_2\{Y\}=\beta^*\{X\}+b_2(\beta^*\{Y\})^2 \tag{7}$$

and:

$$\phi_3\{Y\}=\beta^*\{X\}+b_3(\beta^*\{Y\})^2+c_3(\beta^*\{Y\})^3 \tag{8}$$

Hence:

$$\beta^{**}=A\beta^*\{Y\}+B(\beta^*\{Y\})^2+C(\beta^*\{Y\})^3+\dots \tag{9}$$

where here:

$$A=1+a_2+a_3+\dots$$

$$B=a_2b_2+a_3b_3+\dots$$

$$C=a_3c_3+\dots$$

The constants $A, B, C, \dots$ can be evaluated by substituting β^{**} from Eq (9) for β in Eq (1) for a number of particular values of Y equal to the number of constants, evaluating the terms and integrals analytically or numerically, and solving this set of linear algebraic equations for the constants.

For example, the resulting equation for two terms of Eq (9) at $Y=0$ is:

$$A\beta^*\{0\}+B(\beta^*\{0\})^2=\varepsilon+\frac{(1-\varepsilon)\gamma^2}{2}\int_{-1/2}^{1/2}[A\beta^*\{\xi\}+B(\beta^*\{\xi\})^2]G\{0,\xi\}\mathrm{d}\xi \tag{10}$$

which can be rearranged as:

$$A\left[\beta^*\{0\}-\frac{(1-\varepsilon)\gamma^2}{2}\int_{-1/2}^{1/2}\beta^*\{\xi\}G\{0,\xi\}\mathrm{d}\xi\right]=\varepsilon+B\left[\frac{(1-\varepsilon)\gamma^2}{2}\int_{-1/2}^{1/2}(\beta^*\{\xi\})^2G\{0,\xi\}\mathrm{d}\xi-(\beta^*\{0\})^2\right] \tag{10a}$$

The functions and integrals which were so calculated for collocation points of $Y=0$, 0.25 and 0.50 and parametric values $\varepsilon=0.1$ and $\gamma=0.1$, 0.5 and 1.0, and the resulting constants of Eq (9) are given in Table 1. The corresponding representations of $\beta\{Y\}$ are compared

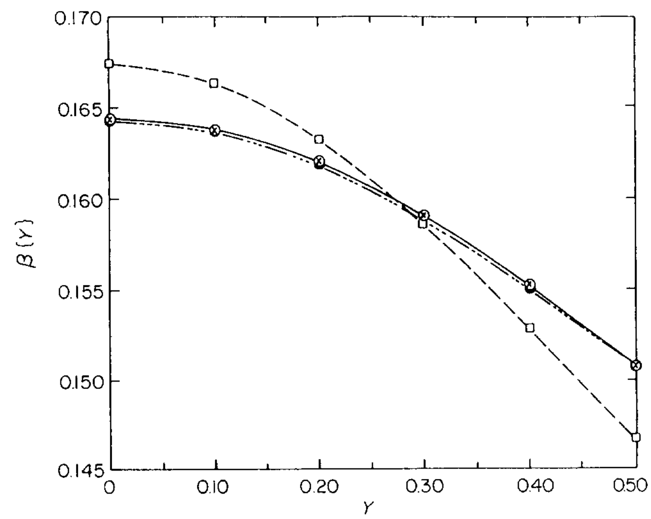

Fig 4 Comparison of approximations and exact solution for dimensionless radiosity on grey parallel strips at equal uniform temperature with $\varepsilon=0.1$ and $\gamma=1.0$. (Symbols as in Fig 2)

Table 1 Functions and coefficients in approximate solutions of Eq (1) for ε=0.1

	γ		
	0.1	0.5	1.0
$\beta^*\{0\}$	0.85123	0.27502	0.16736
$\beta^*\{1/4\}$	0.73458	0.23566	0.16107
$\beta^*\{1/2\}$	0.181083	0.16736	0.14667
$I_1\{0\}$	161.709	1.36389	0.143410
$I_1\{1/4\}$	134.97	1.20358	0.134941
$I_1\{1/2\}$	36.55	0.80685	0.112989
$I_2\{0\}$	134.385	0.33453	0.023025
$I_2\{1/4\}$	98.047	0.28871	0.021645
$I_2\{1/2\}$	16.577	0.18627	0.018083
$I_3\{0\}$	112.1290	0.083654	0.0037026
$I_3\{1/4\}$	72.6938	0.070465	0.0034772
$I_3\{1/2\}$	9.0322	0.04397	0.0028986
One-term			
A	1.0000	1.0000	1.0000
Two-term {0, 1/2}			
A	2.2595	1.50753	1.35297
B	−1.4946	−2.19177	−2.21677
Three-term {0, 1/4, 1/2}			
A	2.13858	1.55315	1.09115
B	−2.89625	−2.64937	1.13598
C	1.63032	1.03977	−10.6725

Here:

$$I_n\{a\} = \int_{-1/2}^{1/2} G\{\alpha,\xi\}(\beta\{\xi\})^n \mathrm{d}\xi$$

with the exact solution in Figs 2–4. The first-order approximation can be considered to be a one-term expansion of Eq (9). The two-term expansion is seen to provide a good representation for $\gamma=0.5$ and 1.0, but to be poorer than the one-term expansion (first-order approximation) for $\gamma=0.1$. The three-term expansion provides an excellent approximation for $\gamma=0.1$ as well. More than three terms could be evaluated, but do not appear to be necessary in this case.

The constants A, B and C were also determined for collocation points of $Y=0$, 0.4 and 0.5. The resulting three-term expression for $\beta^{**}\{Y\}$, for $\varepsilon=0.1$ and $\gamma=0.1$ is compared in Fig 5 with the exact solution and the three-term expression for collocation points of $Y=0$, 0.25 and 0.5. The symmetrical choice of points for collocation is seen to yield a better approximation.

Orthogonal collocation was also tried, but the improvement on the mean for a given number of constants does not appear to be justified by the greater amount of computation to evaluate the constants.

Unequal uniform temperatures

A solution for unequal uniform temperatures can be derived from the results for equal uniform temperatures as follows. The dimensionless formulation for unequal uniform temperatures but the same emissivity is given by the following pair of equations:

$$\beta_1\{Y\} = \varepsilon + \frac{(1-\varepsilon)\gamma^2}{2}\int_{-1/2}^{1/2} \beta_2\{\xi\}G\{Y,\xi\}\mathrm{d}\xi \tag{11}$$

and:

$$\beta_2\{Y\} = \varepsilon\theta_2^4 + \frac{(1-\varepsilon)\gamma^2}{2}\int_{-1/2}^{1/2} \beta_1\{\xi\}G\{Y,\xi\}\mathrm{d}\xi \tag{12}$$

subtracting Eq (12) from Eq (11) gives:

$$\beta'\{Y\} = \varepsilon - \frac{(1-\varepsilon)\gamma^2}{2}\int_{-1/2}^{-1/2} \beta'\{\xi\}G\{Y,\xi\}\mathrm{d}\xi \tag{13}$$

where:

$$\beta'\{Y\} = (\beta_1\{Y\} - \beta_2\{Y\})/(1-\theta_2^4) \tag{14}$$

Eq (13) has the same form as Eq (1) except for the negative sign before the integral, and hence can be solved by exactly the same procedure. Then subtracting Eq (13) multiplied by the factor $(1-\theta_2^4)$ from Eq (11) gives:

$$\beta_1\{Y\} - (1-\theta_2^4)\beta'\{Y\} = \varepsilon\theta_2^4 + \frac{(1-\varepsilon)\gamma^2}{2}\int_{-1/2}^{1/2} \beta'(\xi)G\{Y,\xi\}\mathrm{d}\xi \tag{15}$$

which can be rearranged as:

$$\beta_1''\{Y\} = \varepsilon + \frac{(1-\varepsilon)\gamma^2}{2}\int_{-1/2}^{1/2} \beta_1''\{\xi\}G\{Y,\xi\}\mathrm{d}\xi \tag{15a}$$

where:

$$\beta_1''\{Y\} = \frac{\beta_1\{Y\}}{\theta_2^4 + \frac{1}{\varepsilon}(1-\theta_2^4)\beta'\{Y\}} \tag{16}$$

Eq (15a) is identical in form to Eq (1). Hence Eq (4) provides a first-order approximation for $\beta_1''\{Y\}$, and Eq (9) with the same coefficients can be expected to be a sufficiently accurate approximation for all γ.

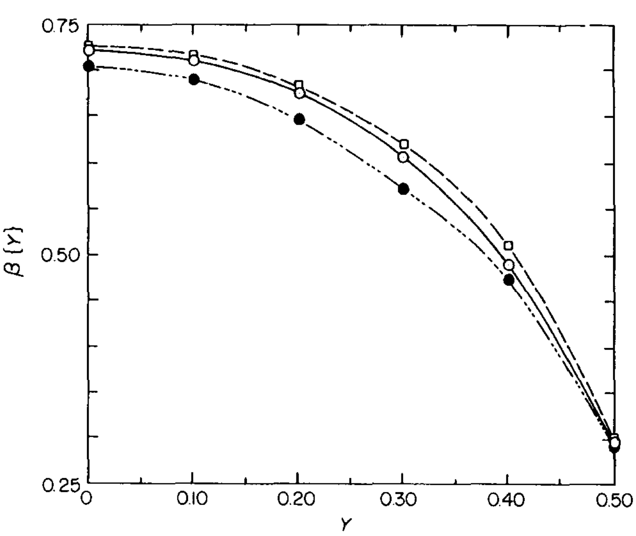

Fig 5 Comparison of approximations and exact solution for dimensionless radiosity on grey parallel strips at equal uniform temperature with $\varepsilon=0.1$ and $\gamma=0.1$. (○ Exact; □ Three-term expansion, Eq (9) with Y=0, 0.25, 0.5; ● Three-term expansion, Eq (9) with Y=0, 0.4, 0.5)

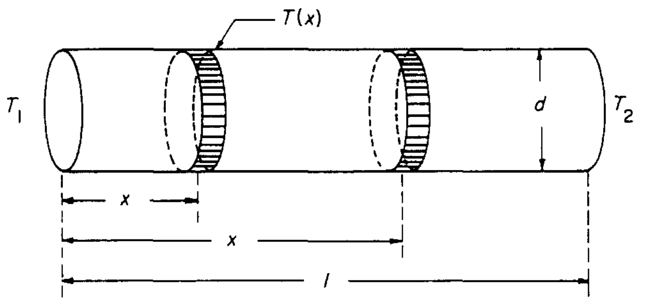

Fig 6 Model for radiative transfer in a grey cylindrical tube

Radiant flux within a cylindrical cavity

As a second geometry consider a round, cylindrical cavity of diameter d and length l (Fig 6). The temperature distribution along the cylindrical wall is a known function of $X=x/d$. The ends of the cylinder at $X=0$ and at $X=l/d=L$ may be considered to be black surfaces at T_1 and T_2, respectively, or open to blackbody surroundings at these temperatures.

Mathematical model

Assuming the same emissivity for all surfaces, the exact formulation for the dimensionless radiosity, from Sparrow and Cess[2] and others, is:

$$\beta\{X\}=\varepsilon\theta^4\{X\}+(1-\varepsilon)\Big[F\{X\}+\theta_2^4F\{L-X\}+\int_0^L\beta\{\xi\}K\{X,\xi\}\mathrm{d}\xi\Big] \quad (17)$$

$F\{X\}$, the fraction of the radiation leaving the disc at $X=0$ which falls on a unit area of the differential ring at X, as given by Hottel[9], is:

$$F\{X\}=\frac{X^2+0.5}{(X^2+1)^{1/2}}-X \quad (18)$$

$F\{L-X\}$, the fraction received from the disc at $X=L$ is obtained from Eq (18) by substituting $L-X$ for X. The fraction of radiation from a differential ring at X' falling on a unit area of the differential ring at X was also shown by Hottel to be:

$$K\{X,X'\}=1-\frac{|X-X'|[(X-X')^2+1.5]}{[(X-X')^2+1]^{3/2}} \quad (19)$$

Although the constant term of Eq (1) has been replaced by a function of X, Eq (17) is still classified as a Fredholm integral equation of the second kind.

Usiskin and Siegel[10] have shown, following Buckley[3], that the above configuration factors can be approximated by the following exponential functions:

$$F\{X\}\simeq\frac{e^{-2X}}{2} \quad (20)$$

$$K\{X,X'\}\simeq e^{-2|X-X'|} \quad (21)$$

Most approximate solutions of Eq (17) have utilized Eqs (20) and (21) but this simplification is not necessary in the method given here.

Direct numerical solution

Eq (17) was first solved by straight-forward numerical integration, using successive iteration, to provide exact solutions for evaluation of the accuracy of the approximations developed below.

First-order approximation

Taking the first term of a series expansion of $\beta\{\xi\}$, as before, and rearranging gives:

$$\beta^*\{X\}=\frac{\varepsilon\theta^4\{X\}+(1-\varepsilon)[F\{X\}+\theta_2^4F\{L-X\}]}{1-(1-\varepsilon)\int_0^L K\{X,\xi\}\mathrm{d}\xi} \quad (22)$$

The integral in Eq (22) can be carried out analytically to obtain:

$$I\{L,X\}=L+1-\frac{2X^2+1}{2(X^2+1)^{1/2}}-\frac{2(L-X)^2+1}{2[(L-X)^2+1]^{1/2}} \quad (23)$$

Second-order approximation

The same process of collocation as used for the parallel-plate problem was applicable using Eq (22) with Eq (23) as the trial function.

Illustrative calculations

These first- and second-order approximation were tested for $\varepsilon=0.21$, $L=25$, $\theta_2=1$ and two wall-temperature distributions.

Linear wall-temperature distribution

The following linear distributions was first investigated:

$$\theta\{X\}=1+\frac{6X}{25} \quad (24)$$

The first-order approximation (Eqs (22) and (23)) is seen in Fig 7 to be in reasonable agreement with the exact solution, except near the peak in the radiosity at $X=23.5$. A two-term collocation at $X=15$ and 23.5 is seen to be in good agreement for all X. Two-term collocations (not shown) at $X=23.5$ and 25, and at $X=0$ and 23.5 were only slightly less accurate overall, but those at $X=0$ and 25, and at $x=15$ and 25 were highly inaccurate and even yielded large negative radiosities. A three-term collocation at $X=0$, 23.5 and 25 is seen in Fig 7 to be somewhat inferior to the two-term collocation at $X=15$ and 23.5. A three-term collocation at $X=15$, 23.5 and 25 (not shown) was better at low X, but worse at high values.

Exponential wall-temperature distribution

The following more complex distribution corresponding to the refractory tube burner experiments of Choi and Churchill[11] was also investigated:

$$\theta\{X\}=1+0.000945e^{0.4479X} \qquad 0\leqslant X\leqslant 18 \quad (25)$$

$$\theta\{X\}=-12.8413+0.076328X^2-7.5171\times10^{-5}X^4 \qquad 18\leqslant X\leqslant 25 \quad (26)$$

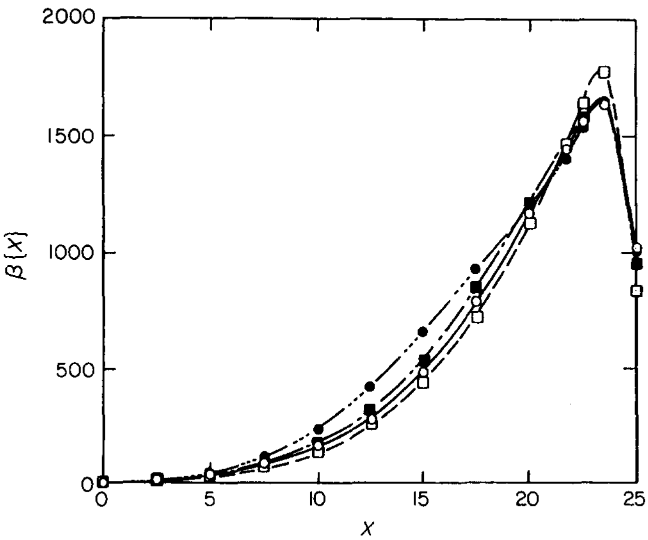

Fig 7 Comparison of approximations and exact solution for dimensionless radiosity on inside surface of a cylinder for $\varepsilon=0.21$, $L=25$, $\theta_2=1$ *and* $\theta\{X\}=1+0.24X$. (○ *Exact;* □ *One-term expansion, Eq (22) with Eq (23);* ■ *Two-term collocation at* $X=15$ *and* 23.5; ● *Three-term collocation at* $X=0$, 23.5 *and* 25)

Again, as indicated in Fig 8, the first-order approximation (Eqs (22) and (23)) gives a good approximation except near the peak in radiosity at $X=22.4$. A two-term collocation at $X=0$ and 22.4 is seen to be in good all-over agreement and to be very accurate for $X>20$. A two-term collocation at $X=22.4$ and 25 was inferior, and those at $X=0$ and 25, 17 and 22.4, and 17 and 25 gave negative radiosities. Three-term collocations at $X=0$, 22.4 and 25 and at $X=17$, 22.4 and 25 also gave highly erratic results.

Interpretation

The above results for the cylinder suggest that collocation with two terms may be more accurate than with three, and that one of the collocation points should be located at the peak in the radiosity as determined by the first-order approximation. The second location may have to be chosen by trial and error to avoid unreasonable behaviour.

Alternative interpretation of first-order approximation

Let Eq (17) be integrated with respect to X from 0 to L:

$$\int_0^L \beta\{X\}\mathrm{d}X=\varepsilon\int_0^L \theta^4\{X\}\mathrm{d}X+(1-\varepsilon)\int_0^L [F\{X\}+$$

$$\theta_2^4F\{L-X\}]\mathrm{d}X+(1-\varepsilon)\int_0^L\left[\int_0^L \beta\{\xi\}K\{X,\xi\}\mathrm{d}\xi\,\mathrm{d}X\right] \tag{27}$$

Since $K\{X,\xi\}$ is symmetrical with respect to X and ξ, the order of integration in the double integral can be inverted (a Dirichlet transformation). Interchange of the dummy variable ξ and X, and rearrangement then yields:

$$\int_0^L\left[\beta\{X\}\left(1-(1-\varepsilon)\int_0^L K\{X,\xi\}\mathrm{d}\xi\right)-(1-\varepsilon)(F\{X\}+\right.$$

$$\left.\theta_2^4F\{L-X\})-\varepsilon\theta^4\{X\}\right]\mathrm{d}X=0 \tag{28}$$

If the outer integrand were independent of its limits, Eq (28) could be satisfied only if the integrand itself vanished for all X. This result again gives Eq (22). Eqs (20) and (21) suggests that the influence of the limits on the integrand may vanish exponentially except for $X=0$ and L. Hence, Eq (22) might be expected to provide a good approximation for large L, except for X near 0 and L. This expectation was confirmed in Figs 7 and 8.

The same procedure can be used with Eq (1) to derive Eq (3). Eq (2) then correctly suggests a greater dependence on the limits of integration, and hence a greater error for the first-order approximation for small γ.

This alternative derivation of the first order approximation is less direct, but identifies the source of error in the approximation (the effect of the limits of the integral) and thus indicates when such an approximation might be useful or not.

Summary and conclusions

This paper has demonstrated how approximate solutions of acceptable accuracy can be developed easily for Fredholm integral equations of the second kind, such as those that arise in radiative transfer between grey surfaces (represented here by Eqs (1) and (17)).

The first-order approximation proposed by Wagner[9], in which the unknown function is represented

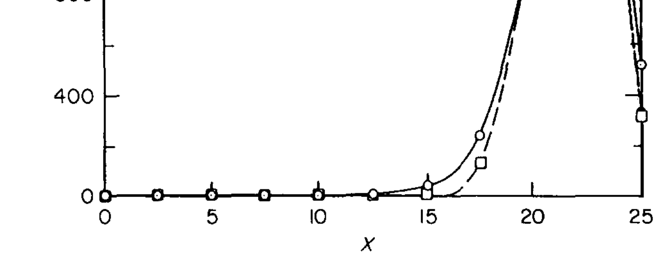

Fig 8 Comparison of approximations and exact solution for dimensionless radiosity on inside surface of a cylinder for $\varepsilon=0.21$, $L=25$, $\theta_2=1$ *and:*

$$\theta=\begin{cases}1+0.0009+5e^{0.4479X} & 0\leqslant X\leqslant 18\\ -12.8413+0.076328X^2-7.517\times10^{-5}X^4, & 18\leqslant X\leqslant 25\end{cases}$$

(○ *Exact;* □ *One-term expansion, Eq (22) with Eq (23);* ● *Two-term collocation at* $X=0$ *and* 22.4) by the first term of a Taylor expansion and thereby taken outside the integral, reduces the process of solution to the simple integration of a view factor, represented here by Eqs (3) and (22). In many cases, as for these examples, this integration can be carried out analytically, yielding an algebraic solution in closed form (Eqs (4) and (22) with (23)).

This simple, first-order approximation may be of sufficient accuracy in many cases. A new, alternative derivation suggests that the error in the approximation arises only from the influence of the limits of integration on the integrand. This influence is apparent, at least qualitatively, from the form of the view factor. Since the view factor usually decays exponentially with distance from the limits of integration, the maximum error can be expected for limited ranges of integration and near the limits of the variable of integration.

A second-order approximation was obtained by simple collocation, using the above first-order approximation as a trial function. This procedure only requires the evaluation of simple integrals of the trial function and configuration factor and the solution of a set of linear algebraic equations equal in number to the number of terms of the collocation. For the first example investigated here, three terms and hence three equations were sufficient to obtain an accurate solution for all parametric values. For the second example, collocation with two terms was more accurate than with three. The accuracy in this case was quite sensitive to the choice of collocation points. Apparently, one of these two points should be at the location of the peak in the radiosity as determined by the first-order approximations. The resulting solutions have the form of an algebraic expression in closed form for the independent variable. However, the constants determined by collocation are specific for the parameters (ε and γ or L).

The earlier approximations mentioned in the introduction differ fundamentally from those demonstrated here in that they reduce the integral equations to a very large set of algebraic equations or to one or more non-linear differential equations. The required amount of computation to solve such equations greatly exceeds that required to evaluate the constants in the second-order approximation developed here. Also, these earlier approximate solutions are in numerical or purely empirical form, whereas the first-order approximation described herein gives a theoretically based dependence in closed form on the independent variable and all the parameters. The second-order approximation is similarly explicit in the independent variable, but not on the parameters.

Although integral equations, such as Eq (1) and (17) which were arbitrarily chosen for illustration here, can be solved directly by numerical integration, the amount of computer time for a given accuracy far exceeds that for the approximate methods. Indeed, the approximate solutions reported here can be carried out on a current state-of-the-art hand-held, programmable computer.

Although Eq (17) was solved by direct numerical integration in the course of this work, a model for combustion which couples this mechanism for thermal radiation with terms for forced convection, thermal conduction and chemical reaction has for 15 years resisted our efforts to obtain a stable and convergent numerical solution. The incorporation of the approximate method developed here has, on the other hand[11], permitted the attainment of such a solution. This approximate method therefore appears to have particular utility in otherwise insoluble problems in which radiative transfer is coupled to other thermal processes.

The examples given here involved one-dimensional radiative transfer. This method of approximation may, by repeated application, have utility in two and three-dimensional radiative transfer, but such applications have not been investigated.

Acknowledgements

This work was sponsored by NSF grants ENG-75-20225 and CPE-82-03967. The use of the computer facilities at the Mobil R&D Corporation, Paulsboro, New Jersey, is gratefully acknowledged.

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